Journal of Computational and Applied Mathematics 233 (2009) 835–841

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Estimation of a Stieltjes function expanded to Taylor series at complex conjugate points

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ARTICLE INFO

Article history:
Received 13 February 2008

MSC:
11J70
41A21

Keywords:
N-point Padé approximants
Stieltjes functions

ABSTRACT

Taylor series expansions of a Stieltjes function f in various complex conjugate points are used to construct the so called unified continued fractions (UCF) terminated on P-th step by a remainder f_P^U named tail of f. We prove that, if f is a Stieltjes function then its tail f_P^U is also a Stieltjes function. The estimations of f are obtained in what follows. Numerical calculations of the new complex bounds on f generated by complex conjugate input data are carried out.

1. Introduction

The Stieltjes S-continued fractions terminated (SCM) by tails f_P^S were amply investigated in literature in the context of an estimation of Stieltjes functions [1–5]. In the present work we construct unified U-continued fractions (UCF) terminated by tails f_P^U. There is a fundamental difference between f_P^U and f_P^S. If a Stieltjes function f is expanded at real points, the tails f_P^U and f_P^S are Stieltjes. However, if these points are complex conjugated then tails f_P^U keep this property, while f_P^S do not. Then, only U-estimates, contrary to S-estimates, are available, if a Stieltjes function is developed at complex conjugate points [6,7,4]. It is worth emphasizing that both S-continued fractions and U-continued ones are special modifications of well known Thiele T-continued fractions, see [1, pp. 127] and [8, pp. 106–108].

2. Regular Stieltjes function and U-continued fraction

Our work is motivated by physical models developed in mechanics of composite materials, where the Stieltjes function f represents some physical parameters. For convenience we assume that $f(-1) = 1$.

Definition 1. The Stieltjes function

$$f(z) := \int_0^1 \frac{d\gamma(u)}{1+zu}, \quad d\gamma \geq 0 \tag{1}$$

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doi:10.1016/j.cam.2009.02.085

taking a finite values in $\mathbb{C} \setminus (-\infty, -1)$ and fulfilling the following condition

$$f(-1) = 1 \tag{2}$$

is called regular Stieltjes function.

Of course, our Stieltjes function can be defined in the cut complex plane $C \setminus (-\infty, -\rho]$ $(\rho > 1)$ and in this case $d\gamma(u) = 0$ for $u \in [1/\rho, 1]$.

Consider the power expansions of f in $2n$ complex points $z_1, z_2, \ldots, z_{2n}$:

$$f(z) = f(z)_{z_j}^{p_j} = \sum_{i=0}^{p_j-1} c_{ij}(z-z_j)^i + O((z-z_j)^{p_j}), \quad j = 1, 2, \ldots, 2n, \tag{3}$$

where $z_{2k-1} = z_{2k}^*$ and $p_{2k-1} = p_{2k}$, $k = 1, 2, \ldots, n$; $c_{ij} = \frac{f^{(i)}(z_j)}{i!}$, $i = 0, 1, \ldots, p_j - 1$. The function f is real symmetric, i.e. it takes complex conjugate values at complex conjugate points: $f(z^*) = [f(z)]^*$. Moreover, the relevant coefficients of power expansions (3) are also complex-conjugated ones. Remark, that for each couple of points, i.e. for z_{2k-1} and its conjugate z_{2k}, we use the same number p_{2k} of informations.

2.1. *U-linear fractional transformation*

First consider the case where all $p_j = 1$. The following consecutive transformations of the function $zf_1(z)$

$$zf_1(z) = z_1f_1(z_1) + \frac{f_1(z_1)(z-z_1)}{1+z\theta_2 f_2(z)}, \ldots, \qquad zf_{2n}(z) = z_{2n}f_{2n}(z_{2n}) + \frac{f_{2n}(z_{2n})(z-z_{2n})}{1+z\theta_{2n+1}f_{2n+1}(z)}, \tag{4}$$

where θ_j are chosen in such a way that $f_j(-1) = 1$, $j = 2, 3, \ldots, 2n+1$ lead to the following UCF expansion terminated by a tail $f_{2n+1}(z)$:

$$zf_1(z) = z_1f_1(z_1) + \cfrac{f_1(z_1)(z-z_1)}{1+\theta_2\left(z_2f_2(z_2) + \cfrac{f_2(z_2)(z-z_2)}{1+\theta_3\left(\cdots + \cfrac{\cdots}{1+\theta_{2n+1}zf_{2n+1}(z)}\right)}\right)}. \tag{5}$$

Let us rewrite (5) as follows

$$\begin{aligned} zf_1(z) &= \left(z_1f_1(z_1) + \frac{f_1(z_1)(z-z_1)}{1+\theta_2}\right)_\times \left(z_2f_2(z_2) + \frac{f_2(z_2)(z-z_2)}{1+\theta_3}\right)\cdots_\times \\ &\times \left(z_{2n}f_{2n}(z_{2n}) + \frac{f_{2n}(z_{2n})(z-z_{2n})}{1+\theta_{2n+1}}\right)_\times zf_{2n+1}(z). \end{aligned} \tag{6}$$

or shortly

$$zf_1(z) = \bigvee_{k=1}^{2n}\left(z_kf_k(z_k) + \frac{f_k(z_k)(z-z_k)}{1+\theta_{k+1}}\right)_\times zf_{2n+1}(z). \tag{7}$$

For arbitrary p_j the general $2n$-point form of UCF is:

$$\begin{aligned} &zf_1(z) = \bigvee_{k=1}^{2n}\left(\bigvee_{j=P_{k-1}+1}^{P_k}\left(z_kf_j(z_k) + \frac{f_j(z_k)(z-z_k)}{1+\theta_{j+1}}\right)\right)_\times zf_{P_{2n}+1}(z), \\ &P_0 = 0, \qquad P_k = \sum_{i=1}^{k} p_i, \quad k = 1, 2, \ldots, 2n. \end{aligned} \tag{8}$$

Now we are able to prove the main theorem.

Theorem 1. *Let f_1 be a regular Stieltjes function defined by*

$$f_1(z) := \int_0^1 \frac{\mathrm{d}\gamma_1(u)}{1+zu} \tag{9}$$

then the tail f_{2n+1} in

$$zf_1(z) = \bigvee_{k=1}^{2n}\left(z_kf_k(z_k) + \frac{f_k(z_k)(z-z_k)}{1+\theta_{k+1}}\right)_\times zf_{2n+1}(z) \tag{10}$$

is a regular Stieltjes function given by

$$f_{2n+1}(z) = \int_0^1 \frac{\mathrm{d}\gamma_{2n+1}(u)}{1+zu}. \tag{11}$$

Proof. Without loss of generality it suffices to prove Theorem 1 for the tail f_3 in

$$zf_1(z) = z_1 f_1(z_1) + \frac{f_1(z_1)(z-z_1)}{1+\theta_2\left(z_2 f_2(z_2) + \frac{f_2(z_2)(z-z_2)}{1+\theta_3 z f_3(z)}\right)}, \tag{12}$$

where $z_2 = z_1^*$ and $f_1(-1) = f_2(-1) = f_3(-1) = 1$. To this end we consider the Padé approximants

$$f_1^K(z) = \sum_{j=1}^{K} \frac{C_{jK}^1}{1+zU_{jK}^1} \quad \text{to } f_1(z) \quad \text{and} \quad f_3^{K-1}(z) = \sum_{j=1}^{K-1} \frac{C_{j(K-1)}^3}{1+zU_{j(K-1)}^3} \text{ to } f_3(z) \tag{13}$$

satisfying the relations (cf. (12))

$$zf_1^K(z) = z_1 f_1^K(z_1) + \frac{f_1^K(z_1)(z-z_1)}{1+\theta_2\left(z_2 f_2^K(z_2) + \frac{f_2^K(z_2)(z-z_2)}{1+\theta_3 z f_3^{K-1}(z)}\right)}, \tag{14}$$

where $z_2 = z_1^*$ and $f_1^K(-1) = f_2^K(-1) = f_3^{K-1}(-1) = 1$ and (cf. (12))

$$\lim_{K\to\infty} f_1^K(z) = f_1(z) \quad \text{and} \quad \lim_{K\to\infty} f_3^{K-1}(z) = f_3(z). \tag{15}$$

If for any K the coefficients C_{jK}^1, U_{jK}^1 and $C_{j(K-1)}^3$, $U_{j(K-1)}^3$ of the Padé approximants f_1^K and f_3^{K-1} given by (13) are positive then both f_1 and f_3 are Stieltjes functions. Since we are given f_1 it means the positive parameters C_{jK}^1, U_{jK}^1 of f_1^K are known. Our aim is to calculate the coefficients $C_{j(K-1)}^3$ and $U_{j(K-1)}^3$ as functions of C_{jK}^1, U_{jK}^1 from (13) and (14). To this end we substitute in (14) $f_3^{K-1}(z) = 0$. We obtain the equation for roots $z_{jK}^{r,3}$ of $f_3^{K-1}(z)$:

$$f_1^K(z) = \frac{C}{1+Uz}, \; C = \frac{f_1^K(z_1) f_1^K(z_2)(z_1-z_2)}{z_1 f_1^K(z_1) - z_2 f_1^K(z_2)}, \; U = -\frac{f_1^K(z_1) - f_1^K(z_2)}{z_1 f_1^K(z_1) - z_2 f_1^K(z_2)} \tag{16}$$

giving three roots $z_{1K}^{r,3}$, $z_{2K}^{r,3}$, $z_{KK}^{r,3}$ immediately

$$z_{1K}^{r,3} = z_1, \qquad z_{2K}^{r,3} = z_2, \qquad z_{KK}^{r,3} = -\infty. \tag{17}$$

The substitution in (12) $f_3^{K-1}(z) = \infty$ yields the equation for poles $z_{jK}^{p,3}$ of $f_3^{K-1}(z)$:

$$f_1^K(z) = \frac{B}{z} + D, \qquad B = -\frac{z_1 f_1^K(z_1) z_2 - z_2 f_1^K(z_2) z_1}{(z_1 - z_2)}, \; D = \frac{z_1 f_1^K(z_1) - z_2 f_1^K(z_2)}{z_1 - z_2}. \tag{18}$$

From (18) two poles $z_{1K}^{p,3}$, $z_{2K}^{p,3}$ are available at once

$$z_{1K}^{p,3} = z_1, \qquad z_{2K}^{p,3} = z_2. \tag{19}$$

Remark 1. It is worth noting, that right hand sides of Eqs. (16) and (18) take at points z_1 and z_2 the same values.

Now we are prepared to investigate the properties of the coefficients C, U and B, D. To this end in (16) and (18) we substitute

$$z_1 = a + ib, \qquad z_2 = a - ib, \qquad f_1^K(z_1) = c - \mathrm{i}d, \qquad f_1^K(z_2) = c + \mathrm{i}d, \quad bd > 0. \tag{20}$$

We obtain (cf. (16))

$$\sum_{j=1}^{K} \frac{C_{jK}^1}{1+zU_{jK}^1} = \frac{C}{1+Uz}, \qquad C = \frac{\left(d^2+c^2\right)}{D}, \; U = \frac{d}{bc-ad}, \tag{21}$$

and (cf. (18))

$$\sum_{j=1}^{K} \frac{C_{jK}^1}{1+zU_{jK}^1} = \frac{B}{z} + D, \qquad B = \frac{(a^2+b^2)d}{b}, \; D = \frac{bc-ad}{b}. \tag{22}$$

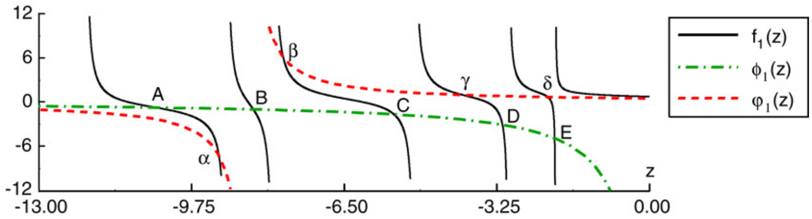

Fig. 1. The intersection A, B, C, D, E, of $f_1(z)$ with $\phi_1(z) = \frac{B}{z} + D$ indicate the positions of poles of f_3 and these of $f_1(z)$ with $\varphi_1(z) = \frac{C}{1+Uz}$ the position $\alpha, \beta, \gamma, \delta$ of zeros of f_3. Note that poles and zeros of f_3 interlace each other.

The coefficients D (computed from $(13)_1$, (20), $(21)_3$) and U (calculated from $(13)_1$, (20), $(22)_3$) take the forms

$$D = \sum_{j=1}^{K} \frac{C_{jK}^1}{\left(1 + aU_{jK}^1\right)^2 + \left(bU_{jK}^1\right)^2}; \qquad U = \sum_{j=1}^{K} \upsilon_{jK} U_{jK}^1, \quad \upsilon_{jK} = \frac{\frac{C_{jK}^1}{\left(1+aU_{jK}^1\right)^2+\left(bU_{jK}^1\right)^2}}{\sum_{j=1}^{K} \frac{C_{jK}^1}{\left(1+aU_{jK}^1\right)^2+\left(bU_{jK}^1\right)^2}}. \tag{23}$$

The relations (23) lead to the inequalitities

$$C > 0, \qquad U_{1K}^1 < U < U_{KK}^1; \quad B > 0, \ D > 0. \tag{24}$$

The inequality $(24)_4$ is also reported in [9]. Due to Remark 1 the difference

$$\delta(z) = \left(\frac{C}{1+Uz}\right) - \left(\frac{B}{z} + D\right) \tag{25}$$

fulfil the relations

$$\delta(z) < 0 \quad \text{if} \ -\infty < z < U \quad \text{and} \quad 0 < \delta(z) \ \text{if} \ U < z < \infty. \tag{26}$$

Because of (24)–(26) the remaining roots $z_{jK}^{r,3}$, $j = 3, 4, \ldots, K$ (cf. (16) or (21)) and the remaining poles $z_{jK}^{p,3}$, $j = 3, 4, \ldots, K+1$ (cf. (18) or (24)) interlace each other, i.e. (cf. (Fig. 1))

$$z_{j(K-1)}^{p,3} < z_{jK}^{r,3} < z_{jK}^{p,3} < -1, \quad j = 4, 5, \ldots, K+1. \tag{27}$$

Due to (17), (19) and (27) f_3^{K-1} has the following rational representation

$$f_3^{K-1}(z) = W_3^{K-1} \frac{(z-z_1)(z-z_2)\prod_{j=3}^{K}(z - z_{jK}^{r,3})}{(z-z_1)(z-z_2)\prod_{j=3}^{K+1}(z - z_{jK}^{p,3})} = \sum_{j=1}^{K-1} \frac{C_{j(K-1)}^3}{1 + zU_{j(K-1)}^3}, \qquad f_3^{K-1}(-1) = 1. \tag{28}$$

From (27) and (28), it follows: the coefficients $C_{j(K-1)}^3$ and $U_{j(K-1)}^3$ forming $f_3^{K-1}(z)$ are positive

$$0 < C_{j(K-1)}^3 \quad \text{and} \quad 0 < U_{j(K-1)}^3 \leq 1, \quad j = 1, 2, \ldots, K-1. \tag{29}$$

These inequalities complete the proof. □

Example 1. In order to illustrate the relations (29) let us consider the Stieltjes function f_1 in the form of sum of simple fractions

$$f_1(z) = \frac{0.05676}{1+0.5z} + \frac{0.170275}{1+0.3333z} + \frac{0.198654}{1+0.2z} + \frac{0.141896}{1+0.125z} + \frac{0.113517}{1+0.1111z} + \frac{0.0851378}{1+0.0833z}. \tag{30}$$

For $z_1 = -15 + \mathrm{i}10$ and $z_2 = -15 - \mathrm{i}10$ it takes values

$$\begin{aligned} f_1(-15+\mathrm{i}10) &= \frac{-10167916445687}{49169252479069} - \mathrm{i}\frac{15727092494267}{49169252479069} \\ f_1(-15-\mathrm{i}10) &= \frac{-10167916445687}{49169252479069} + \mathrm{i}\frac{15727092494267}{49169252479069}. \end{aligned} \tag{31}$$

Table 1
Coefficients of the complex continued fraction expansion of the Stieltjes function f_1 represented by the Stieltjes series (37).

	K			
	1	2	3	4
$f_k(z_k)$	0.2761−i0.0466	0.2302+i0.2861	0.2180−i0.0528	0.2180+i0.05279
θ_{k+1}	0.2761−i0.0466	0.2302+i0.2861	0.2180−i0.0528	0.2180+i0.05279

Solving (12) we obtain

$$f_3(z) = \frac{0.0991466}{1+0.4954z} + \frac{0.259169}{1+0.3197z} + \frac{0.229438}{1+0.1839z} + \frac{0.025962}{1+0.1186z} + \frac{0.101237}{1+0.0952z}. \tag{32}$$

One can check that the inequalities (29) are satisfied.

2.2. Allowed range of values of a Stieltjes function

Let us denote simply by f_P the tail f_{2n+1} in Theorem 1. The function $F_1(z, u)$ bounding the tail $f_P(z)$ was computed in [6, Chapter 17, Eq. 17.16, $R = 1$] and in a different way in [4, Section 1.6, Eq. 1.161, $f_1(-1) = 1$]:

$$F_1(z, u) = \begin{cases} u + 1, & -1 \leq u \leq 0, \\ \dfrac{1-u}{1+zu}, & 0 \leq u \leq 1. \end{cases} \tag{33}$$

By replacing in (8) $f_P(z)$ by $F_1(z, u)$ and $f_1(z)$ by $F_P(z, u)$ we come to the bounding function $zF_P(z, u)$

$$zF_P(z, u) = \bigvee_{k=1}^{2n} \bigvee_{j=P_{k-1}+1}^{P_k} \left(z_k f_j(z_k) + \frac{f_j(z_k)(z - z_k)}{1 + \theta_{j+1}} \right)_{\times} zF_1(z, u), \tag{34}$$

determining the complex bounds $\phi_P(z)$ on a Stieltjes function $f_1(z)$

$$\phi_P(z) = \{w \in \mathbb{C} : w = F_P(z, u); \ -1 \leq u \leq 1\} \tag{35}$$

surrounding the allowed range $\Phi_P(z)$ of values of a Stieltjes function $f_1(z)$ called the inclusion region of $f_1(z)$.

3. Applications of *U*-continued fractions

In order to demonstrate practical calculations by means of UCF we chose as a test function the following Stieltjes one

$$f_1(z) = \frac{\sqrt{z+1} - 1}{z}, \qquad f_1(-1) = 1. \tag{36}$$

The one term Stieltjes series $f_1(z)_{z_j}^{p_j}$, $j = 1, 2, 3, 4$ of f_1 expanded at complex points z : $-5 + i3$, $-5 - i3$, $9 + i7$, and $9 - i7$ are

$$\begin{aligned} &f_1(z) = 0.230 - i0.286 + O(z + 5 - i3); &&f_1(z) = 0.230 + i0.286 + O(z + 5 + i3); \\ &f_1(z) = 0.218 - i0.052 + O(z - 9 - i7); &&f_1(z) = 0.218 + i0.053 + O(z - 9 + i7). \end{aligned} \tag{37}$$

Starting from $(36)_2$ and (37) we evaluate the boundaries $\phi_P(z_0)$ at $z_0 = -15 - i30$ for $P = 1, 3, 5$ (i.e. we use one, three and five informations about f_1).

3.1. Case of one information used

Here the only one value $f_1(-1) = 1$ is known. By (35) we have

$$\phi_1(z_0) = \{w \in \mathbb{C} : w = F_1(z_0, u); \ -1 \leq u \leq 1\}. \tag{38}$$

The inclusion region $\Phi_1(z_0)$ enclosed by boundary $\phi_1(z_0)$ are drawn in Fig. 2.

3.2. Case of three informations used

Three informations used are: $f_1(-1) = 1$ and two first Stieltjes series (37). Hence, on account of (34) one obtains (cf. Table 1)

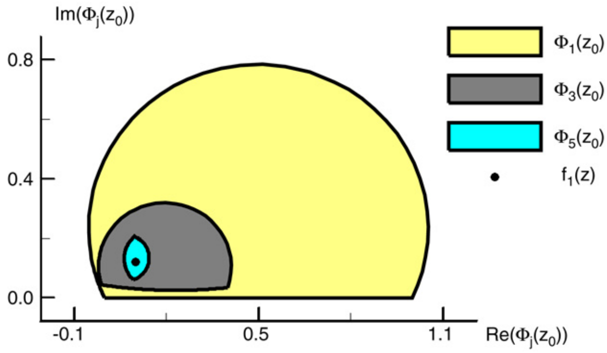

Fig. 2. Sequence of inclusion regions $\Phi_1(z_0)$, $\Phi_3(z_0)$, $\Phi_5(z_0)$ of a function $f_1(z)$ calculated by means of U-continued fraction to $f_1(z)$.

$$zF_3(z,u) = \bigvee_{k=1}^{2}\left(\bigvee_{j=P_{k-1}+1}^{P_k}\left(z_k f_j(z_k) + \frac{f_j(z_k)(z-z_k)}{1+\theta_{j+1}}\right)\right)_{\times} zF_1(z,u), \tag{39}$$
$$P_0 = 0, \qquad P_1 = 1, \qquad P_2 = 2.$$

The function $F_3(z,u)$ leads to the boundary $\phi_3(z_0)$

$$\phi_3(z_0) = \{w \in \mathbb{C} : w = F_3(z_0,u); -1 \le u \le 1\}, \tag{40}$$

enclosing the inclusion region $\Phi_3(z_0)$, see Fig. 2.

3.3. Case of five informations used

Five informations used are: the equality $f_1(-1) = 1$ and four Stieltjes series (37). On account of (34) we obtain (cf. Table 1)

$$zF_5(z,u) = \bigvee_{k=1}^{4}\left(\bigvee_{j=P_{k-1}+1}^{P_k}\left(z_k f_j(z_k) + \frac{f_j(z_k)(z-z_k)}{1+\theta_{j+1}}\right)\right)_{\times} zF_1(z,u), \tag{41}$$
$$P_0 = 0, \qquad P_1 = 1, \qquad P_2 = 2,, \qquad P_3 = 3, \qquad P_4 = 4.$$

The inclusion region $\Phi_5(z_0)$ surrounded by the boundary $\phi_5(z_0)$ computed from

$$\phi_5(z_0) = \{w \in \mathbb{C} : w = F_5(z_0,u); -1 \le u \le 1\} \tag{42}$$

is drawn in Fig. 2.

4. Discussion of the results

4.1. T- and U-continued fractions

Let us rewrite from (12) the U-continued fraction to $zf_1(z)$

$$zf_1(z) = z_1 f_1(z_1) + \frac{(z-z_1)}{\left(\frac{1+\theta_2 z_2 f_2(z_2)}{f_1(z_1)}\right) + \frac{(z-z_2)}{\left(\frac{f_1(z_1)}{\theta_2 f_2(z_2)} + \frac{\theta_3 f_1(z_1)}{\theta_2 f_2(z_2)} z\, f_3(z)\right)}}, \tag{43}$$

where $z_2 = z_1^*, f_1(-1) = f_2(-1) = f_3(-1) = 1, f_2(z)$ is given by (4) and from [8, pp. 107] the T-continued fraction

$$zf_1(z) = z_1 f_1(z_1) + \frac{(z-z_1)}{\rho(z_1 z_2) + \frac{(z-z_2)}{\rho_2(zz_1z_2) - z_1 f_1(z_1)}}, \tag{44}$$

where $z_2 = z_1^*, f_1(-1) = 1,\ \rho(z_1z_2) = \frac{z_2-z_1}{f_1(z_2)-f_1(z_1)}$. From (43) and (44), it follows that

$$\rho(z_1z_2) = \frac{1+\theta_2 z_2 f_2(z_2)}{f_1(z_1)}, \qquad \rho_2(zz_1z_2) = z_1 f_1(z_1) + \frac{f_1(z_1)}{\theta_2 f_2(z_2)} + \frac{\theta_3 f_1(z_1)}{\theta_2 f_2(z_2)} z\, f_3(z) \tag{45}$$

transform T-continued fraction to U-continued one. This fact was observed by the second author. He proved in a different way, that $f_3(z)$ determining $\rho_2(zz_1z_2)$ via $(45)_2$ is a Stieltjes function [9].

4.2. S- and U-continued fractions

Consider now S- and U-fractional expansions of a Stieltjes function $f_1(z)$ to the tails $\check{f}_3(z)$ and $\check{f}_3(z)$, respectively:

$$zf_1(z) = \frac{zf_1(z_1)}{1+\frac{(z-z_1)\hat{\theta}_2\hat{f}_2(z_2)}{1+(z-z_2)\hat{\theta}_3\hat{f}_3(z)}} \quad \text{and} \quad zf_1(z) = z_1f_1(z_1) + \frac{f_1(z_1)(z-z_1)}{1+\check{\theta}_2\left(z_2\check{f}_2(z_2)+\frac{\check{f}_2(z_2)(z-z_2)}{1+\check{\theta}_3 z\,\check{f}_3(z)}\right)} \tag{46}$$

$$z_2 = z_1^*, \qquad f_1(-1) = \hat{f}_j(-1) = \check{f}_j(-1) = 1, \quad j = 2, 3.$$

In order to show the difference between the tails $\hat{f}_3(z)$ and $\check{f}_3(z)$ we consider a normalized Stieltjes function

$$f_1(z) = \frac{\frac{4}{35}}{1+\frac{1}{2}z} + \frac{\frac{8}{35}}{1+\frac{1}{3}z} + \frac{\frac{12}{35}}{1+\frac{1}{5}z}. \tag{47}$$

By substituting $z_1 = 1 + \mathrm{i}$ and $z_2 = 1 - \mathrm{i}$ to (46) we obtain

$$\hat{f}_3(z) = \frac{0.38372 - \mathrm{i}0.01676}{1+(0.26125+\mathrm{i}0.00323)z} + \frac{0.26488+\mathrm{i}0.00941}{1+(0.44889+\mathrm{i}0.00383)z} \quad \text{and}$$

$$\check{f}_3(z) = \frac{0.35707}{1+0.26626z} + \frac{0.27959}{1+0.45535z}. \tag{48}$$

Note that S-tail $\hat{f}_3$ is not a Stieltjes function, while U-tail $\check{f}_3(z)$ is a Stieltjes one. For this reason the Baker's S-continued fraction method [10,6] does not work with complex coefficients of Stieltjes series, while our method of U-continued fractions does work. It is worth adding, that for real coefficients of Stieltjes series both S- and U-methods yield identical estimations of Stieltjes functions.

5. Conclusion

Starting from the Stieltjes series expansions of f_1 in various complex conjugate points we construct U-continued fractions terminated by a tail f_p^U. We prove that, if f_1 is a Stieltjes function then the tail f_p^U is also a Stieltjes function. This result is applied to estimate the function f_1 expanded to a Stieltjes series with complex conjugate coefficients. The numerical calculations of the complex bounds of f_1 generated by the complex conjugate input data are carried out, cf. Fig. 2.

The U-continued fraction method developed in this paper is especially suited for various implementations as a fast, accurate, numerical algorithm, since it is simply recursive and does not involve the solution of a large number of equations or the computation of the zeros of high degree polynomials. It can be applied to any problems of physics, chemistry, mathematics and engineering, if the investigated quantities are expressible in terms of Stieltjes functions. Some of these applications are discussed in the books [6,1,11,2].

Acknowledgements

The authors are grateful to CNRS and PAN for the financial support of this work within the framework of the French-Polish Project 20596. The first author was also supported by the Ministry of Science and Higher Education (Poland) through the Grants No 4 T07A 053 28.

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